CGPG-00/11-2
ICN-UNAM-00/14
gr-qc/0011081

Hairy Black Holes, Horizon Mass and Solitons

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Abstract

Properties of the horizon mass of hairy black holes are discussed with emphasis on certain subtle and initially unexpected features. A key property suggests that hairy black holes may be regarded as 'bound states' of ordinary black holes (without hair) and colored solitons. This model is then used to predict the qualitative behavior of the horizon properties of hairy black holes, to provide a physical 'explanation' of their instability and to put qualitative constraints on the end point configurations that result from this instability. The available numerical calculations support these predictions. Furthermore, the physical arguments are robust and should be applicable also in more complicated situations where detailed numerical work is yet to be carried out.

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I. INTRODUCTION

In the Einstein-Maxwell-Klein-Gordon theory, static black holes are remarkably simple objects. They are completely characterized by their mass and (electric and/or magnetic) charge, *evaluated at infinity*. (For reviews, see [1,2]). As is well-known, this simplicity does not persist once non-Abelian gauge fields are incorporated [3–5]. For example, for a sufficiently large value of the ADM mass $M_{\rm ADM}$, the Einstein-Yang-Mills theory admits distinct spherical, static black hole solutions, labelled by an integer n, all with zero charge at infinity[1] (see Fig. 1). Although in terms of conserved quantities at infinity these solutions are indistinguishable from the Schwarzschild black hole of mass $M_{\rm ADM}$, in the interior their geometries and Yang-Mills fields are quite different from one another. Even the horizon structure varies as we pass from one n to another; for instance, for fixed $M_{\rm ADM}$, the horizon area is a decreasing function of n. Thus, the rigidity tying conserved quantities at infinity to the structure in the interior does not extend to the non-Abelian context. Following the common terminology, if this failure occurs we will say that the theory admits 'hair' and refer to these solutions as 'hairy black holes'.

Of central importance to this paper is the notion of the horizon (or black hole) mass $M_{\rm hor}$ in general (Einstein-matter fields) theories with hair. In the Einstein-Maxwell theory, for static black hole solutions one can set $M_{\rm hor} = M_{\rm ADM}$. However, outside these *globally* time-independent situations, this identification is not viable. For, there may be gravitational radiation in a region far removed from the black hole, and hence irrelevant for any reasonable notion of the horizon mass $M_{\rm hor}$, which would nonetheless obviously contribute to $M_{\rm ADM}$. Since hairy black holes are known to be unstable, such dynamical situations arise naturally in their study. Furthermore, as discussed in Section III A, *even in the static context*, $M_{\rm ADM}$ is a poor measure of the mass of these black holes. Therefore, it is desirable to have a general, systematic approach to the problem of defining horizon mass. In this paper, we will reach this goal by using the recent *isolated horizon* framework, based on Hamiltonian methods [9–11].

Closely related to hairy black holes are the solitonic solutions to Einstein-Yang-Mills equations [12,13,7]. Indeed, it was the Bartnik-McKinnon [12] discovery of these solitons that shook the then prevailing view that there is no qualitative difference between Einstein-Maxwell and Einstein-Yang-Mills theories and paved the way for discovery of hairy black holes. The purpose of this paper is to use the isolated horizon framework to gain new, qualitative insights in to the relation between colored black holes and their solitonic analogs and to point out certain subtleties that arise as we move away from the Einstein-Maxwell context and admit increasing complexity in the hair. A key property of the horizon mass suggests that a colored black hole can be regarded as a 'bound state' of an ordinary, uncolored black hole and a colored soliton. This physical picture turns out to be very useful. It

[1] An infinite family of static spherical solutions was shown to exist in [6,7]. Furthermore, unlike in the Einstein-Maxwell theory, there also exist *non-spherical* static black hole solutions. Although one does not yet have a full control on the entire static sector, the known family of static solutions with zero charge at infinity can labelled by the horizon area and *two* integers [8].

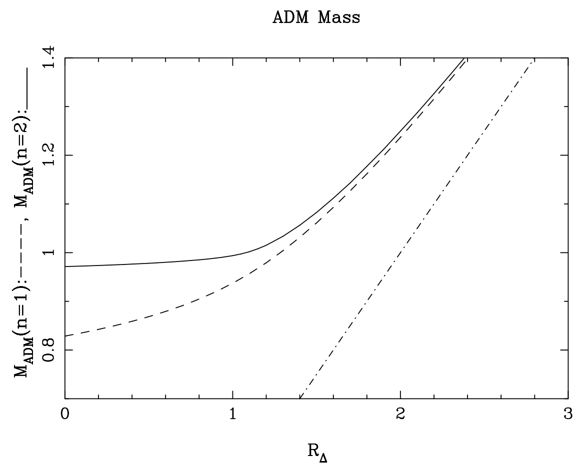

FIG. 1. The ADM mass as a function of the horizon radius R_Δ of static spherically symmetric solutions to the Einstein-Yang-Mills system (in units provided by the Yang-Mills coupling constant). Numerical plots for the colorless, $n = 0$, and the first two families, $n = 1, 2$ of colored black holes are shown. (Note that the y-axis begins at $M = 0.7$ rather than $M = 0$.)

enables one to regard the total (ADM) mass as a sum of three terms: the mass of a bare (i.e. uncolored) black hole, the mass of the appropriate soliton and the gravitational binding energy between the two. Physically expected properties of the gravitational binding energy (such as its negative sign) then determine the qualitative behavior of the horizon attributes of colored black holes (such as surface gravity). These predictions of the model have already been borne out by numerical calculations in simple cases. We will also use the model to argue that instability of colored black holes 'stems' from that of solitons and constrain the final outcome of small perturbations. Finally, as is well-known, quasi-local definitions of mass in general relativity tend not to possess all the properties one is accustomed to from field theories in flat space. We will see that while the isolated horizon framework provides an internally consistent viewpoint and strategy, this very consistency implies that the resulting horizon mass has certain counter-intuitive features. They turn out to have important physical implications, in particular to the stability properties of magnetically charged Reissner-Nordström solutions.

The transition from Abelian to non-Abelian gauge fields is accompanied by the emergence of new scales with dimensions of energy (or length). The classical Einstein-Maxwell theory has no such scale. The coupling constant g of the Einstein-Yang-Mills theory provides an energy scale $M_0 = 1/(g\sqrt{G})$ (in the c=1 units) which signals the onset of new branches (labelled by n) of spherical static solutions (see Fig. 1). In the Einstein-Yang-Mills-Higgs theory, the mass and the self-coupling constant of the Higgs field provide additional scales. Typically, as the number of such scales increases, the departure from the Einstein-Maxwell behavior becomes more significant and the number of unexpected, subtle features grows.

In Section II we recall the definition of the horizon mass in the Einstein-Maxwell theory. In Section III we consider Einstein-Yang-Mills black holes, discuss a key relation between the horizon mass of static, hairy black holes and the mass of the corresponding solitons, introduce the heuristic model of colored black holes mentioned above and derive some of its predictions. In particular, we will show that the notion of the horizon mass can be

used to 'explain' why the magnetically charged Reissner-Nordström solutions are unstable in the Einstein-Yang-Mills theory [14] although they are stable in the Einstein-Maxwell theory [15]. For definiteness, in this section we will restrict ourselves to Einstein-Yang-Mills theory. However, modulo certain subtleties, most considerations of this section will apply also to hairy black holes in more general theories. In Section IV we consider such theories and discuss the relevant subtleties. These arise from the fact that, while in the Einstein-Yang-Mills theory hairy black holes belong to *disjoint* families carrying discrete labels (e.g., integers), in more general theories these families intersect. Section V provides a brief summary.

II. ISOLATED HORIZONS AND HORIZON MASS IN EINSTEIN-MAXWELL THEORY

To define the horizon mass, it suffices to consider the so-called 'weakly isolated horizons'. The precise definition and a detailed discussion can be found in [11] but will not be needed here. In essence, a weakly isolated horizon Δ is an expansion-free, null, 3-dimensional sub-manifold Δ of the 4-dimensional space-time (M, g_{ab}), equipped with an equivalence class of future-directed null normals ℓ^a which Lie-drag (certain components of) the intrinsic connection on Δ.[2] The notion extracts from the definition of a Killing horizon just those properties *local to the horizon* that are needed to obtain a consistent Hamiltonian framework, define the horizon energy, angular momentum and other charges, and prove the zeroth and first laws [11,16]. In the non-rotating case now under consideration, the vector field ℓ^a on Δ plays the role of the static Killing field. However, as explicit examples and general constructions show, the space-time need not admit a Killing field in *any* neighborhood of Δ [17,11].

To define the horizon mass M_Δ, one proceeds as follows. Recall that, in the standard asymptotically flat context, the ADM energy $E^t_{\rm ADM}$ arises as the boundary term at spatial infinity in the expression of the Hamiltonian generating evolution along an asymptotic time-translation t^a. Now the idea is to use as phase space the sector of general relativity consisting of space-times which are asymptotically flat at spatial infinity *and* admit an isolated horizon Δ as *internal boundary.* The Hamiltonian generating an appropriate time-translation then has two surface terms, one at infinity, $E^{(t)}_\infty$, and one at the internal horizon boundary, $E^{(t)}_\Delta$. As one might expect, the first is precisely the ADM energy $E^t_\infty = E^t_{\rm ADM}$. The second, $E^{(t)}_\Delta$, is interpreted as the horizon energy.

However, in the detailed implementation of this strategy a subtlety arises which will play an important role in this paper. At infinity, all metrics in the sector under consideration approach a fixed Minkowskian metric η_{ab}. Therefore, we can simply fix a time-translation

[2]Here, ℓ^a is equivalent to $\tilde\ell^a$ if an only if $\tilde\ell^a = c\ell^a$ where c is a positive constant on Δ. Because the Maxwell stress-energy condition has the property that $-T_{ab}\ell^b$ is future-directed, the Raychaudhuri equation equation implies that the requirement that (Δ, ℓ^a) be expansion-free is equivalent to the requirement that ℓ^a Lie-drags the intrinsic (degenerate) metric on Δ.

Killing field of η_{ab} and demand that t^a agree with that vector field in a neighborhood of infinity. At the internal boundary, by contrast, the geometry is not universal. Therefore, given a candidate time-translation near Δ for one space-time in our sector, it is not a priori clear how to choose the *same* time translation in *another* space-time (also belonging to the sector). To compute the Hamiltonian, one must fix this ambiguity, i.e., choose appropriately a vector field t^a for each space-time in the phase space. In the non-rotating case, it is natural to set $t^a = c\ell^a$ on Δ. However, while c is constant on Δ in any one space-time, it may vary from one space-time to another, i.e., can be a function (of the horizon parameters) on the full phase space. Put differently, in the numerical relativity terminology, one is led to allow *live* evolution vector fields t^a (or, equivalently, lapse-shift pairs that depend on the point on phase space) on space-time.

Let us fix such a live vector field t^a. The evolution along it defines a vector field X_t on the whole phase space. It is then natural to ask if this evolution is Hamiltonian, i.e., if it preserves the symplectic structure. Somewhat surprisingly, the answer is in the affirmative *if and only if* there exists a function $E^{(t)}_\Delta$ of the horizon area a_Δ and electric charge Q_Δ such that

$$\delta E^{(t)}_\Delta \,\hat{=}\, \frac{\kappa_{(t)}}{8\pi G}\,\delta a_\Delta + \Phi_{(t)}\delta Q_\Delta\,. \tag{1}$$

for arbitrary tangent vectors δ to the phase space [11]. Here, $\hat{=}$ stands for equality restricted to Δ, the surface gravity $\kappa_{(t)}$ is defined by t^a via $t^a\nabla_a t^b \,\hat{=}\, \kappa_{(t)} t^b$, and $\Phi_{(t)}$ is the electric potential defined by t^a on Δ. ($\kappa_{(t)}$ is called 'surface gravity' because $t^a \,\hat{=}\, c\ell^a$, with c a constant on Δ. The potential $\Phi_{(t)}$ is unimportant for the purposes of this paper.) Note that (1) implies that $\kappa_{(t)}$ and $\Phi_{(t)}$ are also functions only of a_Δ and Q_Δ; they can not depend on other properties of the solution under consideration.

If $E^{(t)}_\Delta$ is interpreted as the horizon energy, (1) is precisely the first law of black hole mechanics! Thus, the necessary and sufficient condition for the evolution generated by a live vector field t^a to be Hamiltonian is precisely that the first law hold. A live vector field satisfying this condition is said to be *permissible.* The Hamiltonian $H^{(t)}$ generating evolution along a permissible vector field t^a is given by

$$H^{(t)} = \int_\Sigma (\text{constraints})\, \mathrm{d}^3x + E^{(t)}_{\mathrm{ADM}} - E^{(t)}_\Delta \tag{2}$$

where Σ is any partial Cauchy slice extending from the isolated horizon Δ to spatial infinity.

One can now ask if such live vector fields exist on the whole phase space.[3] The answer is in the affirmative [11]. Choose any (regular) function κ_0 of a_Δ and Q_Δ. Then, one can show that there exists a permissible live vector field t^a with $\kappa_{(t)} = \kappa_0$. Thus, in fact, there exists an infinite family of permissible live vector fields, each defining a horizon energy $E^{(t)}_\Delta$ and leading to a first law. While the phase space meaning $E^{(t)}_\Delta$ is transparent, for a general

[3]Since the volume term in the expression (2) is a linear combination of constraints, only the boundary values of the live evolution vector field t^a matter on 'on shell'. Therefore, while discussing evolution vector fields, we will be concerned only with their boundary values.

permissible vector field t^a, its space-time significance is not so clear. Is there perhaps a canonical choice for the boundary value of t^a on Δ? In the Einstein-Maxwell theory, the answer is in the affirmative. Let us demand that the required live vector field t_0^a be such that in *each* static (i.e. Reissner-Nordström) solution t_0^a coincides with that static Killing field which is unit at infinity. This is a natural choice. Furthermore, it fixes $\kappa(t_0)$ uniquely as a function of a_Δ and Q_Δ. Therefore, although we fixed the normalization of t_0^a *only* on the static solution, since there exists a unique static black hole solution for each (a_Δ, Q_Δ), the functional form of $\kappa_{(t_0)}$ is completely fixed on the whole phase space. This in turn selects the boundary value of t_0^a on Δ and determines $E_\Delta^{(t_0)}$ everywhere on the phase space.

It is then natural to set the black hole mass

$$M_{\rm hor} = E_\Delta^{(t_0)} \tag{3}$$

for any space-time in the phase space. This definition of $M_{\rm hor}$ has several pleasing features. First, because t_0 is permissible, (in an obvious notation) the corresponding first law (1) takes the familiar form

$$\delta M_{\rm hor} = \left(\frac{\kappa}{8\pi G}\right)\, \delta a_\Delta + \Phi \delta Q_\Delta\,, \tag{4}$$

but now on the *entire* phase space, including non-static space-times admitting gravitational and electromagnetic radiation. Second, if the evolution vector field coincides with a static Killing field, general phase space considerations imply that the total Hamiltonian is constant on each connected component of the static sector. Since in the Einstein-Maxwell theory, there is no natural energy scale, this constant is necessarily zero. Eq (2) now implies that on static solutions, $M_{\rm hor} = M_{\rm ADM}$. Finally, in presence of radiation, assuming that the horizon extends all the way to future time-like infinity i^+ and that the structure of i^+ is 'the same as that in Reissner-Nordstörm space-times', one can show [9],

$$M_{\rm ADM} - M_{\rm hor} = E_{\mathcal{I}^+}\,, \tag{5}$$

the total energy radiated across future null infinity, $\mathcal{I}^+$. (Equivalently, even in presence of radiation, $M_{\rm hor}$ is the future limit of the Bondi energy as one approaches i^+ along $\mathcal{I}^+$.) Thus, in the Einstein-Maxwell theory, the horizon mass $M_{\rm hor}$ is defined everywhere on the phase space and has the properties one intuitively expects.

Let us summarize. At infinity, $M_{\rm ADM}$ is the norm of the asymptotic 4-momentum. Because the horizon is in a strong field region, we do not have a 4-dimensional translation group on Δ, whence there is no well-defined notion of horizon 4-momentum. Therefore, we need an alternate extra structure to define $M_{\rm hor}$. The idea is to use a preferred time-translation t_0^a and identify $M_{\rm hor}$ with the horizon energy $E_\Delta^{(t)_0}$ defined by this translation. This task could be carried out because, thanks to the no-hair theorems, we could choose a preferred, live evolution field t_0^a for *all* space-times in the full phase space and use it to introduce a canonical definition of $M_{\rm hor}$ on the whole phase space.

III. YANG-MILLS HAIR, HORIZON MASS AND STABILITY

This section is divided in to four parts. In the first, we recall the known results from [10,11]. In the second, we introduce a model of colored black holes as bound states of the

ordinary black holes and appropriate solitons and show that it predicts all the qualitative features of the horizon structure of colored black holes in equilibrium. In the third part, we use the model to discuss instability of colored black holes and put constraints on the final configurations resulting from small perturbations. In the fourth part, we focus on the 'embedded Abelian black holes' and show that the horizon mass of these black holes is higher if they are regarded as Yang-Mills black holes than it is if they are regarded as Einstein-Maxwell black holes. Thus, because the horizon-mass is a phase-space rather than space-time notion, in this case it turns out to be 'theory dependent'. This is a striking and, at a first glance, surprising property. It has an important physical consequence, briefly noted in [10]: using considerations of the first three sub-sections, we will show in detail that it 'explains' why these solutions are unstable in the non-Abelian sector although they are stable in the Einstein-Maxwell theory. This is another striking example of the usefulness of the notion of horizon mass.

A. Horizon mass of colored black holes

Let us now consider the Einstein-Yang-Mills theory with gauge group $SU(2)$. In this case the Reissner-Nordström solutions provide a 2-parameter family of static, spherically symmetric, Abelian black holes, labelled by $(M_{\rm ADM}, Q)$, where Q is now associated with a fixed $U(1)$ subgroup of $SU(2)$. In the Maxwell case, if the potential A_a is a connection 1-form on a trivial bundle, the magnetic charge vanishes identically (and, on non-trivial bundles, it is quantized). In the Yang-Mills case, by contrast, there is a 1-parameter family of magnetically charged, static solutions (with a fixed magnetic charge P_0) *even on the trivial $SU(2)$ bundle* [18]. These are called 'embedded Abelian solutions' because they are isometric to a family of magnetically charged Reissner-Nordström solutions and the isometry maps the Maxwell field strength to the Yang-Mills field strength. The only difference is in the form of the connection. We will return to these black holes in the subsection III D. Of direct interest to this subsection are the families of 'genuinely' non-Abelian, static solutions. Each family is parametrized by one continuous parameter (a_Δ). An infinite number of families, labelled by two integers, is known to exist of which spherically symmetric solutions constitute a sub-class labeled by one integer of Fig. 1 [6,7].

We will now show that, in the Einstein-Yang-Mills theory, the ADM mass fails to be a good measure of the black hole mass *even in the static sector*. Consider, for instance, the branch of spherical, static black holes labelled by $n \neq 0$ (Fig. 1). Let us decrease the horizon area along this branch. In the zero area limit, the solution is known to converge point-wise to a regular, static, spherical solution, representing a Einstein-Yang-Mills *soliton* [12,18]. This solution has, of course, a non-zero ADM mass $M_{\rm ADM}^{\rm sol}$, which equals the limiting value of $M_{\rm ADM}^{\rm BH}$. However, in this limit, there is no black hole *at all*! Hence, this limiting value of the ADM mass can not be meaningfully identified with any horizon mass. By continuity, then, $M_{\rm ADM}^{\rm BH}$ can not be taken as an accurate measure of the horizon mass for any black hole along any $n \neq 0$ branch.

Can one use the isolated horizon framework to define $M_{\rm hor}$ of static black holes in the $n \neq 0$ families? The answer is in the affirmative but there is a subtlety. Considerations of Section II leading to eqs (1) and (2) go through also in this case [11]. Permissible vector

fields can be constructed as in Section II and, for each permissible vector field t^a, we again have a well-defined notion of the horizon energy $E_\Delta^{(t)}$ on the entire phase space, and the associated first law. However, since the no hair theorem now fails, we can no longer single out a *preferred* live vector field t_0^a on the *full* phase space by using static solutions and set $M_{\rm hor} = E_{\rm hor}^{t_0}$. We could focus just on the electrically charged, static Abelian solutions. Surface gravity $\kappa^{\rm Abel}$ of the properly normalized Killing field of these solutions does provide us, as in the Maxwell case, with a specific function $\kappa^{\rm Abel}(a_\Delta, Q_\Delta)$ and we can again use it to obtain a preferred, permissible, live vector field $t^a_{\rm Abel}$ and corresponding horizon energy $E_\Delta^{\rm Abel}$. For these static solutions, we can again set $M_{\rm hor} = E_\Delta^{\rm Abel}$. However, $\kappa^{\rm Abel}$ fails to agree with the surface gravity of the properly normalized Killing field in hairy static solutions. Hence there is no reason to interpret $E_\Delta^{\rm Abel}$ as the horizon mass of the hairy solutions. To summarize, because the uniqueness theorem fails —i.e., because of the hair— we can not single out a canonical function κ_0 of the horizon parameters which agrees with the surface gravity of the (properly normalized) Killing field on *all* static solutions. Therefore, the Einstein-Maxwell prescription for defining the horizon mass $M_{\rm hor}$ on the *full* phase space does not extend to theories with hair.

However, there *does* exist a natural strategy to assign a horizon mass to any *static* solution [11]. For definiteness, let us consider the 1-parameter family of static, spherical solutions labelled by n_0. (However, our discussion and conclusions apply to other static, hairy black holes as well.) Since they have zero electric charge, along this branch the surface gravity (defined by the properly normalized Killing field) is a function $\kappa_{(n_0)}(a_\Delta)$ only of the area a_Δ. By setting $\kappa_0 = \kappa_{(n_0)}$ in the construction outlined in the last sub-section, we obtain a permissible, live vector field $t^a_{n_0}$ on the entire phase space. The corresponding horizon energy $E_\Delta^{(n_0)}$ is given by [10,11]:

$$E_\Delta^{(n_0)} = \frac{1}{2G}\int_0^{R_\Delta} \beta_{(n_0)}(r)\, {\rm d}r\,, \tag{6}$$

where R_Δ is the horizon radius (i.e., the horizon area is given by $a_\Delta = 4\pi R_\Delta^2$), and $\beta_{(n_0)}(R_\Delta) = (2R_\Delta)\,(\kappa_{(n_0)}(R_\Delta))$. While this horizon energy $E_\Delta^{(n_0)}$ has no obvious space-time interpretation for *general* space-times, for static solutions on the branch $n = n_0$, it is natural to set

$$M_{\rm hor}^{(n_0)}(R_\Delta) = E_\Delta^{(n_0)}(R_\Delta) = \frac{1}{2G}\int_0^{R_\Delta} \beta_{(n_0)}(r){\rm d}r\,. \tag{7}$$

We will do so. Note that $M_{\rm hor}^{(n_0)}(R_\Delta)$ vanishes at $R_\Delta = 0$ (and then monotonically increases with area). Hence this definition of the horizon mass is free of the problem faced by the ADM mass, discussed above.

Finally, for any n one can relate the horizon mass $M_{\rm hor}^{(n)}$ to the ADM mass of static black holes. Recall first that general Hamiltonian considerations imply that the total Hamiltonian (2) is constant on every connected component of static solutions (provided the evolution vector field t^a agrees with the static Killing field everywhere on this connected component) [9,11]. In the Einstein-Maxwell case, there is only one connected component and there is no energy scale in the theory. Therefore, we could conclude $H^{(t_0)} = 0$ on all static solutions. In the Einstein-Yang-Mills case, by contrast, there is an infinite number of connected components of static solutions *and* there is a natural energy scale, $M_0 = 1/g\sqrt{G}$. Therefore,

on each distinct connected component, in principle, the Hamiltonian could be a *different* multiple of M_0. This is exactly what happens. Since the Hamiltonian is constant on any n-branch, we can evaluate it at the solution with zero horizon area. This is just the soliton, for which the horizon area a_Δ, and the horizon mass $M_{\rm hor}$ vanish. Hence (2) implies $H^{(t_0,n)} = M_{\rm sol}^{(n)}$. Thus, we conclude:

$$M_{\rm sol}^{(n)} = M_{\rm ADM}^{(n)} - M_{\rm hor}^{(n)} \tag{8}$$

on the entire nth branch [10,11]. Thus, the ADM mass contains two contributions, one attributed to the black hole horizon and the other to the outside 'hair', captured by the 'solitonic residue'.

B. A physical model of colored black holes

Considerations of Section III A lead to a heuristic, but quite powerful physical model of a colored black hole. In this subsection we will develop this model and derive some of its consequences on the equilibrium properties of static black holes.

Recall that for static black holes in the Einstein-Maxwell theory, the future limit $M_{\rm Bondi}^{i^+}$ of the Bondi mass along $\mathcal{I}^+$ equals the horizon mass $M_{\rm hor}$. What is the situation for the non-Abelian black holes of the Einstein-Yang-Mills theory? In the zero area limit along any $n \neq 0$ branch, one is left with a soliton, i.e., with a regular bound state which persists all the way to i^+ and thus contributes to the mass at i^+. Because the solution is static, the ADM mass, the Bondi mass and the mass at i^+ are all equal. As we move away from zero area, a true black hole appears. For small areas, (7) implies that $M_{\rm hor}^{(n_0)}$ is small. But, in relation, $M_{\rm ADM}^{(n_0)}$ can be large because 'there is still a solitonic contribution to the ADM mass', quantified in (8). In the Einstein-Maxwell case, there are no solitons; there is no 'residue' at i^+ apart from the black hole itself. This is why for static black holes, $M_{\rm hor}$ equals $M_{\rm ADM}$ and charges at infinity suffice to characterize the fields on the horizon. In the Einstein-Yang-Mills case, by contrast, the conserved quantities at infinity do *not* in general capture the whole story. For hairy black holes, the structure at $\mathcal{I}^+$ (or i^0) does not rigidly determine the structure at the horizon; there is a solitonic residue at i^+ which accounts for the lack of rigidity. This suggests that we think of a colored black hole as *a 'bound state' of an ordinary, 'bare' black hole and a 'solitonic residue'*. Indeed, we can trivially rewrite the total mass of (8) as a sum of three parts, one referring to the soliton, one to the 'bare' black hole and one to the binding energy:

$$\begin{aligned} M_{\rm ADM}^{(n)}(R_\Delta) &= M_{\rm sol}^{(n)} + M_{\rm hor}^{(0)}(R_\Delta) + (M_{\rm hor}^{(n)} - M_{\rm hor}^{(0)})(R_\Delta) \\ &= M_{\rm sol}^{(n)} + M_{\rm hor}^{(0)}(R_\Delta) + E_{\rm binding}(R_\Delta)\,, \end{aligned} \tag{9}$$

where $M_{\rm hor}^{(0)}(R_\Delta) = R_\Delta/2G$ is the horizon mass of the Schwarzschild black hole of radius R_Δ. Thus, one can say that the effect of the solitonic residue is to 'dress' the black hole mass from its 'bare value' $M_{\rm hor}^{(0)}(R_\Delta)$ to $M_{\rm hor}^{(n)}(R_\Delta)$. The fact that the total space-time mass $M_{\rm ADM}$ is just a sum $M_{\rm ADM} = M_{\rm hor}^{(n)} + M_{\rm sol}^{(n)}$, without an explicit interaction term, can be understood as follows. Recall that, in general relativity, the total Lagrangian or the Hamiltonian is just

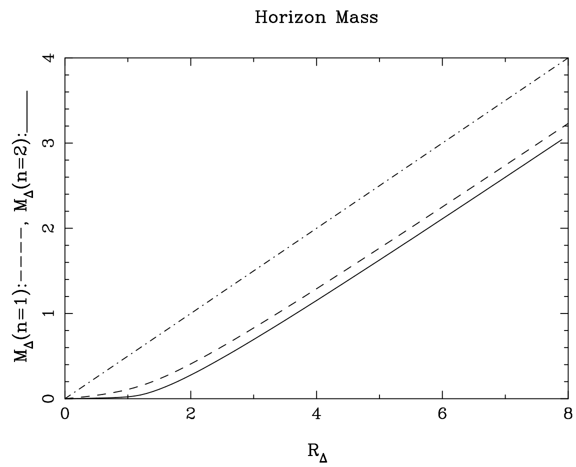

FIG. 2. The horizon mass $M_{\rm hor}$ as a function of the radius R_Δ (in units provided by the Yang-Mills coupling constant). Numerical plots for $n = 0, 1, 2$ spherical, colored black holes in the Einstein-Yang-Mills theory are shown.

a sum of the gravitational part without matter and the matter contribution. There is no *explicit* interaction term between matter and gravity; the interaction is 'absorbed' in the description by simply replacing the flat metric in the 'bare' matter terms by the curved, physical metric, making the full expression covariant. In an analogous way, the interaction between the soliton and the black hole is expressed covariantly by simply replacing the 'bare' black hole mass $M^{(0)}_{\rm hor}(R_\Delta)$ by the 'dressed' mass $M^{(n)}_{\rm hor}(R_\Delta)$ as in (8).

Let us adopt this model and work out some of its implications. More precisely, the idea is to use this model to derive qualitative features of the properties of colored black holes assuming, whenever necessary, properties of colored solitons. Since the colored black holes solutions have been obtained numerically, we do not have simple analytical expressions of their horizon attributes such as surface gravity and horizon mass. Furthermore, typically such quantities have been computed *only for low values of* n. In the remainder of this sub-section we will show that all the qualitative features of the known numerical plots of these quantities as functions of n and horizon radius R_Δ can be derived from our model. Furthermore our predictions apply to *all* values of n and R_Δ.

First, since $E_{\rm binding}$ is the gravitational binding energy between the soliton residue and the bare black hole, it must be negative. Thus, the model together with the elementary physical fact about the gravitational binding energy leads us to the first prediction:

i) $M^{(n)}_{\rm hor}(R_\Delta) < M^{(0)}_{\rm hor}(R_\Delta)$ *for all* $n > 0$ *and all values of the horizon radius* R_Δ.

At first this conclusion seems somewhat counter-intuitive since, for a given horizon area, one normally thinks of the colored black holes as 'excited states' and the $n = 0$, Schwarzschild black hole as the 'ground state'. However, that intuition comes from the properties of the ADM mass: it is known that $M^{(n)}_{\rm ADM}(R_\Delta) > M^{(0)}_{\rm ADM}(R_\Delta)$. Explicit numerical calculations have in fact confirmed i) for low values of n (see Fig. 2): while $M^{(n)}_{\rm ADM}(R_\Delta)$ does grow with n, $M^{(n)}_{\rm sol}$ grows so that i) holds. The model predicts that this inequality will hold for all $n > 0$.

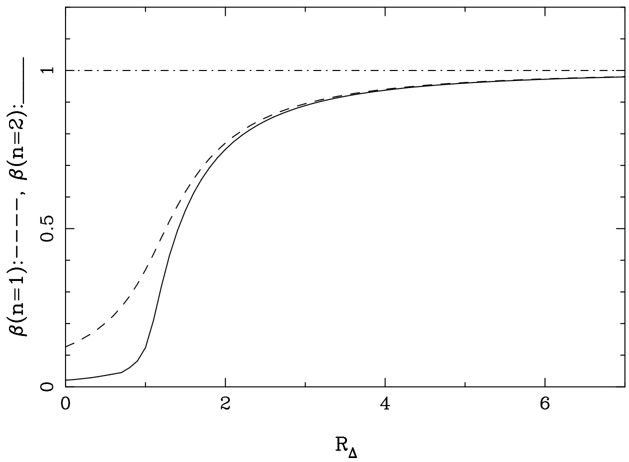

FIG. 3. Numerical plots of $\beta_{(n)} = 2\kappa_{(n)}\, R_\Delta$ as function of the horizon radius R_Δ for $n = 1, 2$ spherical, colored black holes in the Einstein-Yang-Mills theory. For large R_Δ, all curves asymptotically approach the $\beta = 1$ line.

Next, let us use the expression (7) of the horizon mass. From i), we conclude:

$$\int_0^{R_\Delta} \left(\beta_{(n)} - \beta_{(0)}\right)(r)\, \mathrm{d}r < 0$$

for all n and R_Δ. Hence, it follows that $\left(\beta_{(n)} - \beta_{(0)}\right)(R_\Delta) < 0$, whence we conclude

ii) $\left(\kappa_{(n)} - \kappa_{(0)}\right)(R_\Delta) < 0$ *for all* $n > 0$ *and* R_Δ.

Again, this prediction is borne out by detailed numerical simulations for low n (see Fig. 3).

We can now use the physically expected properties of the gravitational binding energy to make sharper predictions. Let us fix R_Δ and vary n. Then, the mass $M^{(0)}_{\rm hor}$ of the bare black hole is fixed but, as is well-known, the soliton mass $M^{(n)}_{\rm sol}$ increases with n. Now, since the absolute value of the gravitational binding energy increases as the mass of either of the two bound objects grows, $\mid E_{\rm binding} \mid$ must increase with n. This implies:

iii) *For any fixed value of* R_Δ *both the horizon mass* $M^{(n)}_{\rm hor}$ *and the surface gravity* $\kappa_{(n)}$ *are monotonically decreasing functions of* n.

As Fig 2 and Fig 3 show, these predictions are borne out numerically for low n.

Next, let us consider the complementary situation and keep $n > 0$ fixed and vary R_Δ. Now the soliton mass $M^{(n)}_{\rm sol}$ is fixed and the bare black hole mass $M^{(0)}_{\rm hor}(R_\Delta) = R_\Delta/2G$ increases with R_Δ. Hence, again (the absolute value of) the gravitational binding energy must increase with R_Δ. Thus, we conclude:

iv) *For any fixed* n, $(R_\Delta/2G - M^{(n)}_{\rm hor})(R_\Delta)$ *is a monotonically increasing function of* R_Δ *whence, from (7),* $\beta_{(n)}(R_\Delta) < 1$ *for all* $n > 0$ *and* R_Δ.

Again, Fig. 2 and Fig. 3 show that this prediction is borne out by numerical studies for low values of n. Furthermore, this property can be directly derived from the definition (7) of

$M_{\rm hor}^{(n)}(R_\Delta)$ and prediction ii) above. Indeed,

$$\left(\frac{R_\Delta}{2G} - M_{\rm hor}^{(n)}\right)(R_\Delta) \;=\; M_{\rm hor}^{(0)} - M_{\rm hor}^{(n)} \;=\; \int_0^{R_\Delta} (\kappa_{(0)} - \kappa_{(n)})(r)\mathrm{d}r$$

and ii) ensures that the integrand is positive.

Next, we can put a lower bound on $\beta_{(n)}$. We will use the fact that, for any fixed n the ADM mass $M_{\rm ADM}^{(n)}(R_\Delta)$ is a monotonically increasing function of R_Δ. Since $M_{\rm sol}^{(n)}$ is fixed on the n-th branch of static solutions, it follows from (8) that

v) $M_{\rm hor}^{(n)}(R_\Delta)$ *is also a monotonically increasing function of* R_Δ. *(7) now implies that, for all* n *and* R_Δ, $\beta_{(n)}$ *and* $\kappa_{(n)}$ *are positive functions of* R_Δ. *Hence* $M_\Delta^{(n)}$ *is also positive for all* n *and* R_Δ, *except for* $R_\Delta = 0$ *where it vanishes.*

Finally, the model has a prediction on the asymptotic behavior (for large R_Δ) of $M_{\rm hor}^{(n)}(R_\Delta)$, $\beta_{(n)}(R_\Delta)$ and $\kappa_{(n)}(R_\Delta)$. Using the known property of the ADM mass, $M_{\rm ADM}^{(n)}(R_\Delta) > M_{\rm ADM}^{(0)}(R_\Delta) = R_\Delta/2G$, we conclude:

$$M_{\rm hor}^{(n)}(R_\Delta) \;=\; M_{\rm ADM}^{(n)}(R_\Delta) - M_{\rm sol}^{(n)} \;>\; \frac{R_\Delta}{2G} - M_{\rm sol}^{(n)}\,.$$

Hence, using prediction i) we conclude that, for any given n, the curve $M_{\rm hor}^{(n)}(R_\Delta)$ lies between the two parallel lines $f_1(R_\Delta) = R_\Delta/2G - M_{\rm sol}^{(n)}$ and $f_2(R_\Delta) = R_\Delta/2G$. Furthermore, since $M_{\rm hor}^{(n)}(R_\Delta)$ is a monotonically increasing function of R_Δ, it follows that:

vi) *For any* n, *the curve showing the dependence of* $M_{\rm hor}^{(n)}(R_\Delta)$ *on* R_Δ *becomes asymptotically parallel to the two lines which bound it, i.e., has slope* $1/2G$ *for large* R_Δ. *Hence, by (7), the curves showing the dependence of* $\beta_{(n)}$ *on* R_Δ *asymptotically approach the curve* $\beta_{(0)}(R_\Delta) = 1$ *and the curves showing the functional dependence of* $\kappa_{(n)}$ *of* R_Δ *asymptotically approach the curve* $\kappa_{(0)}(R_\Delta) = 1/2R_\Delta$.

These six properties account for all qualitative features of Fig. 2 and Fig. 3 i.e., the dependence of $M_{\rm hor}^{(n)}$ and $\beta_{(n)}$ on n and R_Δ. In particular for Fig. 3, we have shown that our model implies that the $\beta_{(n)}$ curves: a) intersect the y-axis at distinct points between 0 and 1 and lie in the upper right hand quadrant, b) the curves never intersect, c) have the property that higher the n, lower the curve, and d) for large R_Δ become asymptotically tangential to the curve $\beta_{(0)}(R_\Delta) = 1$. To arrive at this qualitative understanding of the horizon properties, in addition to the model, we used only two known qualitative properties of the ADM mass [18]: $M_{\rm sol}^{(n)}$ increases with n and, for any given n, $M_{\rm ADM}^{(n)}$ increases monotonically with R_Δ.

C. Instability of colored black holes

We will now argue that the model also 'explains' the instability of the colored black holes in terms of the instability of the solitons. Under small perturbations, the solitons are unstable; the energy 'stored' in the bound state of the soliton is radiated away to future null infinity, $\mathcal{I}^+$ [19]. In our model, each static black hole with $n \neq 0$ is accompanied by a soliton which 'hovers around all the way to i^+' and carries information about its hair. Therefore, it

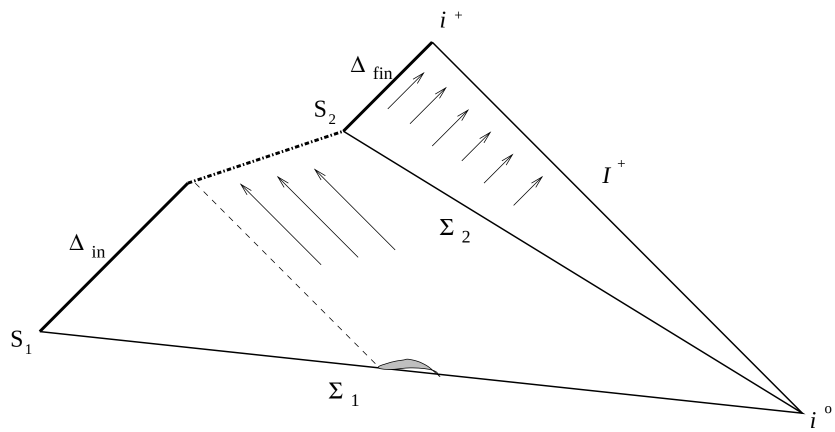

FIG. 4. An initially static colored black hole with horizon $\Delta_{\rm in}$ is slightly perturbed and decays to a Schwarzschild-like isolated horizon $\Delta_{\rm fin}$, with radiation going out to future null infinity $\mathcal{I}^+$.

is natural to expect that under small perturbations in the exterior region, this soliton would be unstable and radiate away across $\mathcal{I}^+$ and Δ. At the end of the process, there should be no solitonic residue at i^+. The end product should thus be an uncolored (i.e. Schwarzschild) black hole whose horizon mass $M^{(0)}_{\rm hor}$ equals the future limit M_{i^+} of the Bondi mass along $\mathcal{I}^+$. Our suggestion that the instability of the colored black hole is due to the instability of the accompanying soliton is supported by the fact [20] that all the colored black holes on the nth branch have the same number (namely, $2n$) of unstable modes as the nth soliton. Of course the detailed features of these unstable modes will differ especially because they are subject to different boundary conditions in the two cases.

In general, however, even if one component of a bound system is unstable, the total system may still be stable if the binding energy is sufficiently large. An obvious example is provided by the deuteron nucleus —a bound state of a proton and a neutron. Therefore, to see if our bound state of a bare black hole and a soliton is in fact unstable, let us examine the energetics. If the decay suggested above is to occur, the total energy in the initial bound state, $E_{\rm initial} = M^{(n)}_{\rm hor}(R^{\rm initial}_\Delta) + M^{(n)}_{\rm sol}$, should exceed the energy in the final black hole $E_{\rm final} = M^{(0)}_{\rm hor}(R^{\rm final}_\Delta)$. Since the area of the final uncolored black hole is at least as large as the area of the initial colored black hole, the decay can occur only if the following inequality holds:

$$M^{(n)}_{\rm hor}(R^{\rm initial}_\Delta) + M^{(n)}_{\rm sol} > M^{(0)}_{\rm hor}(R^{\rm initial}_\Delta)\,. \tag{10}$$

The inequality is in fact satisfied. For, the left side is the ADM mass $M^{(n)}_{\rm ADM}(R_\Delta)$ while the right side is the ADM mass $M^{(0)}_{\rm ADM}(R_\Delta)$ of the uncolored black hole of the same radius $R^{\rm initial}_\Delta$, and the ADM mass is known to increase with n. Thus, the energetics are such that the instability of the soliton can persist also in the bound state, forcing the solitonic residue to radiate away.

Let us therefore assume that the process illustrated in Fig. 4 does take place and examine its consequences. Thus, we assume that the event horizon has an initial isolated component $\Delta_{\rm initial}$ with color n and radius $R^{\rm initial}_\Delta$ and there is a small perturbation on the initial partial

Cauchy surface M. The perturbation evolves and, because of the instability of the soliton, the solitonic residue is radiated away and we are left with a final, uncolored isolated horizon $\Delta_{\rm final}$ with radius $R_\Delta^{\rm final}$ (satisfying $R_\Delta^{\rm final} \geq R_\Delta^{\rm initial}$) and energy $E_{\mathcal{I}^+}$ radiated across future null infinity $\mathcal{I}^+$.[4] The ADM mass of the space-time does not change in this process. Hence neglecting the energy in the initial perturbation (i.e., in the limit 'from above' of a sequence of perturbations with decreasing energy) we have:

$$M_{\rm ADM}^{(n)} = M_{\rm hor}^{(n)}(R_\Delta^{\rm initial}) + M_{\rm sol}^{(n)} = M_{\rm hor}^{(0)}(R_\Delta^{\rm final}) + E_{\mathcal{I}^+}\,. \tag{11}$$

Using (9) and the expression of the ADM mass of the uncolored (i.e. Schwarzschild) black hole in terms of its horizon radius, this equality can be re-expressed as

$$\left[M_{\rm sol}^{(n)} - \mid E_{\rm binding}^{\rm initial} \mid\right] = \frac{1}{2G}\left(R_\Delta^{\rm final} - R_\Delta^{\rm initial}\right) + E_{\mathcal{I}}^{+}\,. \tag{12}$$

Thus, the 'available energy' for the process is given by:

$$E_{\rm avail}^{(n)} = M_{\rm sol}^{(n)} - \mid E_{\rm binding}^{\rm initial} \mid\,. \tag{13}$$

Note that $E_{\rm avail}^{(n)}$ can be computed *knowing just the initial configuration.* However, the distribution of this energy in to a part which goes in to increasing the horizon area and the part which gets radiated to null infinity will depend on the details of the initial perturbation.

Our model enables us to make further qualitative predictions about this process. Note first that, since for a fixed n, the binding energy $\mid E_{\rm binding}^{\rm initial} \mid$ is a monotonically increasing function of $R_\Delta^{\rm initial}$ and $M_{\rm sol}^{(n)}$ is independent of $R_\Delta^{(n)}$, the available energy $E_{\rm avail}^{(n)}$ decreases as $R_\Delta^{\rm initial}$ increases. In this sense, for a given n, a larger colored black hole is less unstable than a smaller one. This prediction is supported by the fact that for the $n = 1$ colored black holes, the frequency of all unstable modes is a decreasing function of the area, whence the characteristic decay time grows with area [18,22]. Next, let us fix $R_\Delta^{\rm initial}$ and let n vary. Then, since the available energy can be re-expressed as $E_{\rm avail}^{(n)} = M_{\rm ADM}^{(n)} - M_{\rm ADM}^{(0)}$, it follows that it increases with n. In this sense, for a given $R_\Delta^{\rm initial}$, higher the n, more unstable the colored black hole. This prediction is supported by the fact that the number of unstable modes is given by $2n$. It would be interesting to analyze if there exist further, more detailed correlations.

Finally, it is natural to ask if the horizon radius necessarily grows in this process. Using an additional input, we will now show that the answer is in the affirmative. Recall from [10,11] that each isolated horizon in the Einstein-Yang-Mills theory carries a well-defined magnetic charge P_Δ:

$$P_\Delta = -\frac{1}{4\pi}\oint \mid F \mid \;\epsilon \tag{14}$$

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[4]In a realistic situation, in distant future the event horizon will become isolated only asymptotically and will probably not admit a *finite* isolated piece $\Delta_{\rm final}$. However, our conclusions will continue to hold in this case if the approach to isolation is sufficiently fast.

where the integral is taken over any 2-sphere cross-section of Δ, ϵ is the area 2-form on this cross-section, and $\mid F \mid = [\mathrm{Tr}(F_{ab}\epsilon^{ab})\ (F_{cd}\epsilon^{cd})]^{\frac{1}{2}}$. The additional fact we need is that P_Δ vanishes if and *only* if $n = 0$. Therefore, in the process under consideration, P_Δ necessarily changes. Let us now *assume* $R_\Delta^{\mathrm{initial}} = R_\Delta^{\mathrm{final}}$ and arrive at a contradiction. Since the expansion of the null normal ℓ^a of any event horizon is necessarily non-negative, it follows that the expansion is necessarily zero. The Raychaudhuri equation then implies $T_{ab}\ell^a v^b = 0$ for all vectors v^a tangential to the event horizon and the form of the Yang-Mills stress-energy T_{ab} then implies that the field strength must satisfy $F_{ab}\ell^a v^b = 0$. One can now use the reasoning of [11] (see Eq (VI.7) to Eq(VI.9) of that paper) to conclude that P_Δ is conserved all along the event horizon (even though some of the isolated horizon boundary conditions may be violated in the intermediate region). This is however impossible because while $P_\Delta^{\mathrm{initial}}$ is non-zero $P_\Delta^{\mathrm{final}}$ vanishes because the end product of the process is a Schwarzschild black hole. Hence our assumption $R_\Delta^{\mathrm{initial}} = R_\Delta^{\mathrm{final}}$ can not hold; the area of the event horizon must *necessarily increase* in this process. Hence (12) implies:

$$E_{\mathcal{I}}^{+} = \left(M_{\mathrm{sol}}^{(n)} - \mid E_{\mathrm{binding}}^{\mathrm{initial}} \mid\right) - \frac{1}{2G}\left(R_\Delta^{\mathrm{final}} - R_\Delta^{\mathrm{initial}}\right) < E_{\mathrm{avail}}^{(n)} . \tag{15}$$

There may well exist a more sophisticated versions of this argument which provide stronger bounds on the change in the horizon area in terms of $P_\Delta^{\mathrm{initial}}$ and $R_\Delta^{\mathrm{initial}}$ and hence a better upper bound on the energy that can be radiated away to $\mathcal{I}^+$.

D. Instability of Embedded Abelian solutions

Let us now consider the 'embedded Abelian solutions' with magnetic charge. In the Einstein-Maxwell theory, the electromagnetic connection is defined on a non-trivial $U(1)$ bundle of a fixed Chern-class, corresponding to the fixed value P_0 of the magnetic charge while in the non-Abelian case, the Yang-Mills potential is defined on a trivial $SU(2)$ bundle. (In the non-Abelian case, P_0 has the same dimensions as the inverse $1/g$ of Yang-Mills coupling constant. In the conventions/units generally used in the literature, $P_0 = \pm 1$.) In either case, a detailed analysis shows that Eqs (1) and (2) continue to hold, without any extra terms. Since the presence of an electric charge plays no essential role in this analysis, for simplicity, let us focus on the sector of the phase space where it vanishes. Then, in both cases, we are left with a 1-parameter family of static solutions which can be coordinatized by the value of the horizon area. Using the expression of the surface gravity

$$\kappa_{(P_0)} = \frac{1}{2R_\Delta}\left(1 - \frac{GP_0^2}{R_\Delta^2}\right)$$

of this family, we can introduce a permissible evolution field t_0 and obtain the corresponding first law:

$$\delta E_\Delta^{(t_0)} = \frac{1}{8\pi G}\kappa_{(P_0)}\delta a_\Delta$$

Let us integrate this law to obtain the horizon Energy:

$$E_\Delta^{(t_0)}(R_\Delta) = \frac{R_\Delta}{2G}\left(1 + \frac{GP_0^2}{R_\Delta^2}\right) + C \tag{16}$$

where C is the integration constant. For colored black holes, the constant was determined [10,11] by requiring that $E_\Delta^{(t_0)}$ should vanish in the limit R_Δ goes to zero. For (embedded) Abelian black holes under consideration, by contrast, the minimum value of R_Δ is $\sqrt{G} \mid P_0 \mid$ when the black hole becomes extremal. Therefore, we need a new strategy to determine the constant C.

In the Einstein-Maxwell case, it is natural to appeal to the duality invariance of the theory. The sector of the theory with zero magnetic charge but arbitrary electric charge Q, we know [9,11] that the horizon energy is given by[5]:

$$E_\Delta^{(t_0)}(R_\Delta) = \frac{R_\Delta}{2G}\left(1 + \frac{GQ^2}{R_\Delta^2}\right) \tag{17}$$

Hence, by duality invariance, we conclude $C = 0$.

One could also have reached the same conclusion by starting ab-initio in the dual description, i.e., by considering from the beginning the vector potential for the electric field as the configuration variable and the magnetic field as the momentum variable and allowing the magnetic charge P to be arbitrary but setting the electric charge to zero from the beginning. One could then work on a trivial bundle and, using the dual of the standard framework of [11], conclude that $C = 0$. To summarize, because of our choice of the evolution vector field t_0^a, we can interpret $E_\Delta^{(t_0)}$ as the horizon mass $M_{\rm hor}^{\rm EM}$ of the Einstein-Maxwell magnetically charged black holes and each of the two arguments given above leads us to the result

$$M_{\rm hor}^{\rm EM} = \frac{R_\Delta}{2G}\left(1 + \frac{GP_0^2}{R_\Delta^2}\right) . \tag{18}$$

Let us now turn to Yang-Mills theory. To determine the constant C in (16), we can no longer appeal to duality invariance or use a dual description with the vector potential of the electric field as the configuration variable. Thus, we need a new input. Let us first consider the colored, spherically symmetric, *non-Abelian* black hole space-times discussed in the last sub-section. It is known that as n tends to zero, these colored, *non-Abelian* black hole space-times tend to the magnetically charged, embedded Abelian black hole space-time provided the horizon radius satisfies the bound $R_\Delta \geq \sqrt{G}P_o$ [18]. If $R_\Delta = \sqrt{G} \mid P_0 \mid$, the magnetically charged black hole is extreme. The new input is that as n tends to infinity, $\beta_{(n)}(R_\Delta)$ tends to zero for $0 \leq R_\Delta \leq \sqrt{G}P_0$ (see Fig. 3). Hence, using (7) it follows that as n tends to infinity, keeping $R_\Delta = \sqrt{G}P_0$, the horizon mass $M_{\rm hor}^{(n)}(R_\Delta)$ tends to zero. Hence, in the Yang-Mills phase space, we are led to set the constant C of (16) to: $C = -P_0/\sqrt{G}$. Consequently, in Yang-Mills theory, the horizon mass of the embedded Abelian solutions is given by [10]:

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[5]In the reasoning used in [9,11], a key role is played by the fact Q is allowed to vary, i.e., that the first law reads: $\delta M_\Delta = (\kappa/8\pi G)\delta a_\Delta + \Phi\delta Q$, and by the fact that on this phase space with variable Q, there is no constant of dimensions of energy we can construct from the only available constants G and c.

$$M_{\rm hor}^{\rm EYM} = \frac{R_\Delta}{2G}\left(1 + \frac{GP_0^2}{R_\Delta^2}\right) - \frac{\mid P_0 \mid}{\sqrt{G}}\,. \tag{19}$$

Thus, the horizon mass in the Yang-Mills theory is *lower* than that in the Einstein theory.

It is this difference that accounts for the strikingly different stability properties of these black holes in the two theories. Since the solutions are asymptotically flat at spatial infinity in the standard sense, their ADM mass is determined only by the asymptotic behavior of the space-time metric and is given by:

$$M_{\rm ADM} = \frac{R_\Delta}{2G}\left(1 + \frac{GP_0^2}{R_\Delta^2}\right) \tag{20}$$

in *both* theories. Thus we conclude:

$$M_{\rm ADM} - M_{\rm hor}^{\rm EM} = 0, \quad \text{while} \quad M_{\rm ADM} - M_{\rm hor}^{\rm EYM} = \frac{\mid P_0 \mid}{\sqrt{G}} \tag{21}$$

Thus, as guaranteed by general arguments from symplectic geometry, the difference between the ADM and the horizon mass is indeed constant on the connected component of magnetically charged static solutions but the value of the constant is theory dependent. In the Einstein-Maxwell theory, the value is zero. All the energy in the solution is associated with the horizon and there is no provision in the 'energy budget' for energy to be radiated away. In the Einstein-Yang-Mills theory, the energy budget does have such a provision. The future limit of the horizon mass ($= M_{\rm hor}^{\rm EYM}$) is *less* than the future limit of the Bondi mass along $\mathcal{I}^+$ ($= M_{\rm ADM}$). Therefore, there is a mismatch at i^+. Although in the limit $n \to \infty$ there is no regular soliton,[6] since

$$\lim_{n\to\infty} M_{\rm sol}^{(n)} \neq 0\,,$$

from the standpoint of energetics, there is still a solitonic residue at i^+. Hence, as a limiting case of the situation considered in the last sub-section, our reasoning implies that the embedded Abelian black holes should be unstable in the Einstein-Yang-Mills theory even though their Einstein-Maxwell counterparts are stable. Detailed analysis [14] has shown that this is precisely what happens! Furthermore, we can compute the 'available energy' for the black hole of radius R_Δ in EYM. It has the form $E_{\rm avail}^{\rm EYM} = P_0^2/2R_\Delta$. This suggests that the frequencies of the unstable modes will be smaller for larger black holes.

It may seem surprising at first that there is such a marked difference in the stability properties in spite of the fact that the space-time geometries of these Einstein-Maxwell and Einstein-Yang-Mills black holes are identical. However, since these geometries 'live' in distinct phase spaces, modes which can trigger instability within one phase space can be absent in the other. Indeed, stability is not a property of a specific solution in isolation but refers also to other 'accessible' states and the set of accessible states is different in the two phase spaces. The notion of the horizon mass can capture this difference because it is a *phase space* notion rather than a space-time notion.

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[6]It has been shown that for $r > 1$ the limiting solution approximates the extremal RN solution [23]

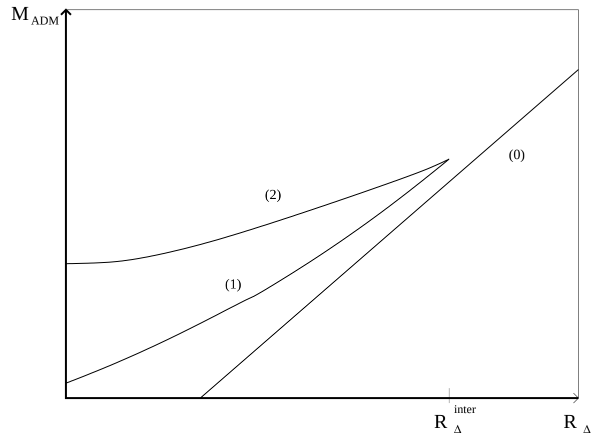

FIG. 5. The ADM Mass as a function of the horizon radius R_Δ in theories with a built-in non-gravitational length scale. The schematic plot shows crossing of families labelled by $n = 1$ and $n = 2$ at $R_\Delta = R_\Delta^{\rm inter}$.

To summarize, these magnetically charged black holes bring out the fact that the horizon mass $M_{\rm hor}$ of an isolated horizon Δ can not be determined knowing only the space-time geometry unless one knows before hand which phase space this space-time belongs to. At first, this 'ambiguity' in the horizon mass seems surprising. However, as we have seen, it has a deep physical implication. In the next section, we will see that a further subtlety associated with the notion of horizon mass arises as we consider theories with additional dimensionful constants and more complicated hair.

IV. HAIRY BLACK HOLES IN MORE GENERAL THEORIES

Let us now turn to more general theories with hair, by allowing Higgs or Procca fields in addition to the Yang-Mills fields, or considering Einstein-Skyrme theories [24,25,18]. These theories have additional dimensionful constants which trigger new phenomena. The static black hole sector of such theories enables one to analyze these novel aspects of the interplay between gravity and the nonlinear matter fields. Of primary interest to us is the 'crossing phenomena' of Fig. 5 where curves in the 'phase diagram' (i.e. plot of the ADM mass versus horizon radius) corresponding to the two families of static solutions intersect. With due attention to this phenomenon, considerations of Section III will extend to hairy black holes in these more general theories. In this section we will discuss the relevant subtleties that must be taken in to account.

We will begin with a qualitative argument to elucidate the 'origin' of the crossing phenomenon. Let us first ignore gravity and consider these theories in Minkowski space-time. The coupling constants and other parameters (such as the vacuum values of fields) in these theories provide a length scale which determines the 'size' $R_{\rm sol}^{\rm Mink}$ of the solitonic solution. Let us now couple these theories to general relativity. Gravitational effects, being attractive, can only reduce this size. Hence, if a black hole is to exhibit any hair at all, its horizon radius R_Δ must be *less than* $R_{\rm sol}^{\rm Mink}$. (In fact, using more detailed arguments involving the

pressure of matter fields near the horizon and at infinity, one can give a sharper bound: $R_\Delta < (2/3) R_{\rm sol}^{\rm Mink}$ [5].) Thus, in these theories which have an in-built length scale already in Minkowski space-time, the horizon radius R_Δ of hairy black holes is bounded above by that length scale which is independent of Newton's constant. Note that, Yang-Mills theory in Minkowski space has no such built-in length scale whence there is no bound on the horizon radius of hairy, static Einstein-Yang-Mills black holes.

If there is an upper bound to the horizon radius, it is natural to ask what happens in the phase diagram to the individual, connected branches corresponding to static black holes. Recall first that we can view the space of static black hole solutions as the manifold of extrema of the ADM mass functional on phase space [21]. In the Einstein-Yang-Mills theory, each connected component of this manifold is a sub-manifold without boundary. Let us suppose that the topology of these sub-manifolds does not change abruptly as the new coupling (and other) constants are slowly switched on to arrive at more general theories. Then, if the horizon radius of the new theories is bounded from above, the sub-manifolds which are disconnected in the Einstein-Yang-Mills theory must merge smoothly in these theories.[7] Then, in the projection of this smooth sub-manifold to the $(R_\Delta, M_{\rm ADM})$ plane, one would see two branches which cross at the point corresponding to the maximum value of R_Δ. This is indeed what is found in numerical investigations of the spherically symmetric, static black hole sectors of Einstein-Skyrme, Einstein-Yang-Mills–Proca, and Einstein-Yang-Mills-Higgs theories [24–26] and, more recently, in the static axially symmetric black hole sector of Einstein-Yang-Mills-Higgs theory [27]. Our discussion suggests that this crossing phenomenon will occur in any theory that contains a length scale independent of G and allows hairy black holes.

Let us start by analyzing the way two branches merge at the cross-over point. For simplicity, let us focus on static black holes that carry no other charge *at infinity* beyond the ADM mass. As long as the energy momentum tensor in the theory satisfies appropriate energy conditions, the static black holes can be assumed to admit a Killing horizon with a bifurcation two surface [28]. For such black holes, arbitrary perturbations are known to obey the first law of black hole dynamics [21]:

$$\delta M_{\rm ADM} = \frac{1}{8\pi} \kappa \, \delta a_\Delta \, . \tag{22}$$

Now consider the phase diagram (i.e., the plot of the ADM mass as a function of the horizon radius) for this family and focus on the intersection point between the two branches. If the two curves intersect at a finite angle, the first order change in the mass for a given infinitesimal change in the horizon area would depend on the branch used to approach the

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[7] In the case of the 'embedded Abelian family', by contrast, the sub-manifold of static solutions has a boundary corresponding to extremal black holes. However, in this case, the phase space itself has a boundary consisting of space-times whose isolated horizons have zero surface gravity and the sub-manifold just ends at this boundary. If sub-manifolds of static *colored* black holes were to terminate without merging, these sub-manifolds would have a true boundary in the *interior* of the phase space.

intersection point. In view of (22), this would contradict the fact that the surface gravity has a well defined value at the solution corresponding to the intersection point. Hence, we conclude:

vi)' at any cross-over point, the two branches must have the same tangent vector.

Explicit calculations have confirmed this behavior in all theories where crossing phenomena has been seen to occur.

Next, let us examine how the crossing phenomenon affects the notion of the horizon mass. As in Section III A, for all theories under consideration, given any permissible evolution vector field t^a we are again led to a horizon energy $E_\Delta^{(t)}$ and the corresponding first law (1). The delicate point again is the selection of a preferred evolution vector field. Because of the presence of hair —more precisely, because the value of the horizon area fails to determine a static black hole uniquely— we can no longer use the strategy that was successful in the Einstein-Maxwell case to define a canonical evolution vector field t_0^a and use $E_\Delta^{(t_0)}$ as the horizon mass M_Δ on the *full* phase space. However, as in the Einstein-Yang-Mills case, we can again focus on any one branch, say the nth, of static solutions, use the surface gravity $\kappa_{(n)}(R_\Delta)$ to select a permissible evolution vector field $t_{(n)}^a$ and construct the horizon energy $E_\Delta^{(n)}$ and set (a la (7)) the mass $M_{\rm hor}^{(n)}$ of any black hole on the nth branch to be:

$$M_{\rm hor}^{(n)}(R_\Delta) = E_\Delta^{(n)}(R_\Delta) = \frac{1}{2G}\int_0^{R_\Delta} \beta_{(n)}(r)\mathrm{d}r\,. \tag{23}$$

This strategy is tenable except at the cross-over points. At these points, there is a problem: Because the function $\beta_{(n)}$ depends on n, the value of the horizon mass is now ambiguous. If branches n and $n+1$ cross, as in Section III B we have $\beta_{(n+1)}(R_\Delta) < \beta_{(n)}(R_\Delta)$, whence $M_{\rm hor}^{(n+1)} < M_{\rm hor}^{(n)}$. We saw in Section III D that, since the notion of the horizon mass refers to the phase space, the value of the horizon mass of a black hole is not determined simply by its space-time geometry. Here we see another, and starker, aspect of the same phenomenon.

In the magnetically charged black holes of Section III D, the same space-time appeared in two different phase spaces and the value of its horizon mass depended on which phase space one used. In the present case, there is a single phase space. However, we can use either the nth family of static solutions and choose a permissible evolution vector field $t_{(n)}^a$ using the functional dependence of the surface gravity $\kappa_{(n)}$ on the horizon radius R_Δ to fix the normalization, or we can do the same using the $(n+1)$th family. Since the functional dependences of $\kappa_{(n)}$ and $\kappa_{(n+1)}$ on the horizon radius are necessarily different, we obtain two distinct vector fields and two distinct Hamiltonians which generate time-evolution along them. Each of the resulting horizon energies $E_\Delta^{(n)}$ and $E_\Delta^{(n+1)}$ is a well-defined function on *the entire phase space* and leads to a first law (1). Although the two energies differ, there is complete consistency because the two energies arise from two different evolutions (i.e., two different normalizations of the evolution vector field on the horizon). The ambiguity arises only when we want to identify these energies with horizon masses and is confined just to the cross-over points. As remarked in the Introduction, definitions of quasi-local mass in general relativity invariably encounter unexpected features and the ambiguity in the horizon mass that arose (in Section III D and) here is the counter-intuitive feature encountered in the present Hamiltonian approach.

However, in spite of —or rather, because of— this ambiguity, one can extract useful information from this definition of horizon mass. Let the nth and $(n+1)$th branch intersect at the horizon radius $R_\Delta^{\rm inter}$. Then, using (8) it follows that:

$$M_{\rm ADM}^{(n)}(R_\Delta) - M_{\rm hor}^{(n)}(R_\Delta) = M_{\rm sol}^{(n)} \quad \text{and} \quad M_{\rm ADM}^{(n+1)}(R_\Delta) - M_{\rm hor}^{(n+1)}(R_\Delta) = M_{\rm sol}^{(n+1)}\,. \tag{24}$$

Hence, using (7), at the intersection point we obtain:

$$M_{\rm ADM}(R_\Delta^{\rm inter}) = M_{\rm sol}^{(n)} + \frac{1}{2G}\int_0^{R_\Delta^{\rm inter}} \beta_{(n)}(r)\,{\rm d}r = M_{\rm sol}^{(n+1)} + \frac{1}{2G}\int_0^{R_\Delta^{\rm inter}} \beta_{(n+1)}(r)\,{\rm d}r \tag{25}$$

The second equality yields:

$$M_{\rm sol}^{(n+1)} - M_{\rm sol}^{(n)} = \frac{1}{2G}\int_0^{R_\Delta^{\rm inter}} \beta_{(n)}(r)\,{\rm d}r - \frac{1}{2G}\int_0^{R_\Delta^{\rm inter}} \beta_{(n+1)}(r)\,{\rm d}r = \frac{1}{2G}\oint \beta(r)\,{\rm d}r \tag{26}$$

where the last closed counter integral is performed by first moving along the nth branch from $R_\Delta = 0$ to $R_\Delta = R_\Delta^{\rm inter}$, then moving back along the $n+1$th branch to $R_\Delta = 0$ and finally sliding down the y axis to the point $R_\Delta = 0$ on the nth branch (see Fig.5).[8] Note that the counter integral is non-zero because $\beta_{(n)}$ and $\beta_{(n+1)}$ are *distinct functions* of R_Δ, or equivalently, *precisely* because the horizon mass undergoes a jump at the cross-over point. This equation is striking in that it provides a quantitative relation between the horizon properties of hairy black holes and masses of solitons, even though the two belong to completely different sectors of the theory. The equality was first checked numerically in certain static, axi-symmetric hairy black hole solutions in the Einstein-Yang-Mills-Higgs theory [27]. Given the generality of our considerations, one expects this relation to hold also for more general matter sources which lead to the crossing phenomena, such as Yang-Mills-Higgs, Yang-Mills-Proca and Skyrme fields.

Finally, our discussion of the qualitative behavior of equilibrium properties (Section III B) and instability of colored Einstein-Yang-Mills black holes (Section III C) will extend also to these more general black holes with two provisos: a) due care is taken of the ambiguity in $M_{\rm hor}$ at the cross-over points; and, b) property vi) in Section III B regarding the asymptotic behavior of horizon quantities for large $R_\Delta^{(n)}$ replaced by vi)' noted above on the behavior of the phase diagrams at the cross-over points.

Remark: The above discussion can be used also to shed light on the behavior of the ADM mass of *solitons* in the *Einstein-Yang-Mills theory*, in particular its dependence on n. Let us begin with the collection of Einstein-Yang-Mills-Higgs theories parametrized by the vacuum expectation value η of the Higgs field. Now, the static sector of these theories consists of extrema of the ADM mass at a fixed area of the internal boundary [21] and, when $\eta = 0$, the extrema are configurations where the Higgs field vanishes identically. Thus, we can consider the static sector of the Einstein-Yang-Mills theory as the limit $\eta \to 0$ of the static sector of

[8] Since the 'y-axis' corresponds to zero area, points on it do not belong to the black hole phase space considered here. Therefore, one should close the contour by moving along the vertical line $R_\Delta = \epsilon$ and then take the limit $\epsilon \to 0$.

the Einstein-Yang-Mills-Higgs theory. Now, in the Einstein-Yang-Mills-Higgs theories, the crossing phenomenon occurs and the masses of solitons are related by Eq. (26) for all values of η. Taking the limit when $\eta \to 0$ we arrive at an identical equation, now relating the ADM masses of the Einstein-Yang-Mills solitons. In this limiting case, the branches do not cross at a finite value of the horizon radius. However, from the discussion at the beginning of this section, one expects that as η tends to zero, the value of the horizon radius at the crossing point tends to infinity. Now, from the Fig. 3, and more generally from prediction vi), we see that $(\beta^{(n)} - \beta^{(n+1)})(r) \to 0$ as $R_\Delta \to \infty$. Thus, in the limit, all the β curves meet at infinity, including the Schwarzschild branch corresponding to $\beta_{(0)}(R_\Delta) = 1$. Hence, it is plausible that the area bounded by the two curves $\beta_{(n)}$ and $\beta_{(n+1)}$ in the plot of β as a function of R_Δ is finite, i.e., that the contour integral on the right side of (26) is well-defined also in the Einstein-Yang-Mills theory. Thus, we obtain an explicit prediction for the value of the ADM mass of the n-th soliton in this theory:

$$M_{\rm sol}^{(n)} = \frac{1}{2G}\int_0^\infty (1 - \beta_{(n)}(r))\,\mathrm{d}r\,. \tag{27}$$

Again, the equation is striking because it enables one to compute *soliton* masses from surface gravity of *black holes*, now in the Einstein-Yang-Mills theory. To our knowledge, this prediction has not been tested. Note, however, that it uses not only the framework presented in this paper, but also a continuity assumption on the embedding of the static sector of the Einstein-Yang-Mills theory in that of the Einstein-Yang-Mills-Higgs theories. Nonetheless, a qualitative consequence of (27) is borne out by numerical simulations. The behavior of $\beta_{(n)}$ in Fig. 3 suggests that $M_{\rm sol}^{(n+1)} - M_{\rm sol}^{(n)}$ approaches zero as n tends to infinity and this behavior is supported by the numerical study of low n soliton solutions [12,18], and by rigorous methods in [7]. It would be interesting to test (27) in a more direct ways. Finally, note that if we combine Eq. (27) with Eq. (8) we arrive at an expression for $M_{\rm ADM}(R_\Delta)$,

$$M_{\rm ADM}^{(n)}(R_\Delta) = \frac{R_\Delta}{2G} + \frac{1}{2G}\int_{R_\Delta}^\infty (1 - \beta_{(n)}(r))\,\mathrm{d}r \tag{28}$$

The first term is what we have called the 'bare mass', and the second term could be interpreted as the 'mass of the hair', corresponding to the soliton mass plus the binding energy as in Eq. (9).

V. DISCUSSION

Because of the black hole uniqueness theorems in Einstein-Maxwell theory [1,2] it was widely believed that black holes would have no hair also in theories more complicated, non-Abelian gauge fields. As soon as this expectation was shown to be incorrect [3,4], the subject of hairy black holes became a focal point in the mathematical physics literature. By now, static hairy black hole solutions have been found numerically for a variety of matter sources coupled to general relativity. Their horizon attributes, such as the dependence of the surface gravity on the horizon area, have been plotted and their stability has been studied in some detail. The literature in the field has grown steadily over the last decade with discoveries of new families of solutions made by combining analytic and numerical techniques. Therefore,

a long list of facts regarding hairy black holes is now available. In other branches of physics, when sufficient data accumulates either through experiments or numerical simulations, the subject is generally considered to be ripe for phenomenology; simple heuristic models are used to put 'order' in the data. In a similar spirit, we introduced a heuristic model of hairy black holes as bound states of ordinary black holes and solitons and used it to account for qualitative features of a significant fraction of the accumulated data.

The notion of the horizon mass played a key role both in motivating the model and in its applications. In particular, this notion could be used effectively to gain qualitative insight in to 'why' hairy black holes are unstable, and more strikingly, 'why' the magnetically charged Reissner-Nordström black holes are unstable in the Einstein-Yang-Mills theory although they are stable in the Einstein-Maxwell theory. Since horizon mass plays an important role in our phenomenology, we also investigated certain subtleties and pointed out some of its counter-intuitive properties. Finally although for definiteness we restricted ourselves to the Einstein-Yang-Mills theory in Section III, our phenomenological model is valid also in theories with more complicated matter sources and our qualitative analysis can be repeated in this more general context provided one takes an appropriate care of the subtleties encountered in Section IV at the cross-over points.

ACKNOWLEDGMENTS

We would like to thank S. Fairhurst for discussions, P. Bizon, P. Forgacs and S.T. Yau for pointing out some references, and U. Nucamendi for help with the figures. This work was supported in part by NSF grants INT9722514, PHY95-14240, PHY00-90091, the Eberly research funds of Penn State, DGAPA-UNAM grant No IN121298, CONACyT grants J32754-E and 32272-E, and by an NSF-CONACyT collaborative grant.

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